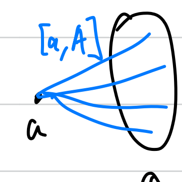
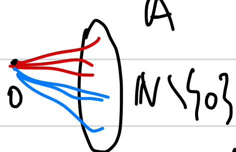
Math Logic: Model Theory & Computability
Lecture 16

Infinite Ramsey Theorem. For any ℓ, k, and any colouring $c: [\mathbb{N}]^\ell \to \{0,\dots,k-1\}$, there is an infinite $M \subseteq \mathbb{N}$ c-monochromatic subset.

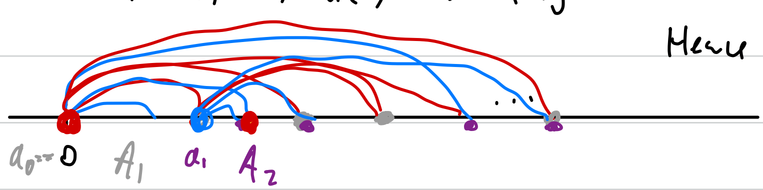
Proof. For $a\in\mathbb{N}$ and $A\subseteq\mathbb{N}$, denote $[a,A] := \{\{a,a_1,\dots,a_{\ell-1}\} : \{a_1,\dots,a_{\ell-1}\}\in[A\setminus\{a\}]^{\ell-1}\}$, so for $\ell=2$, $[a,A]=\{\{a,a'\}: a'\in A\setminus\{a\}\}$.





We inductively build a decreasing sequence (A_n) of infinite subsets of $\mathbb{N}$ such that $[a_n, A_{n+1}]$ is c-monochromatic, where $a_n := (\min A_n)\setminus A_{n+1}$. Let $A_0:=\mathbb{N}$. Suppose $A_n\subseteq\mathbb{N}$ is defined and is infinite, let $a_n := \min A_n$. By the Infinite-in-finite Pigeonhole Principle, there is an infinite $A\subseteq A_n\setminus\{a_n\}$ such that $[a_n, A_{n+1}]$ is monochromatic. Let $A_{n+1}:=A$, let $A_\infty := \{a_n : n\in\mathbb{N}\}$, hence infinite, and define $c': A_\infty\to\{0,1,\dots,k-1\}$ by setting $c'(a_n)$ to be the colour of the set $[a_n,A_{n+1}]$. Again by the Pigeonhole Principle, there is a c'-monochromatic subset $M\subseteq A_\infty$, say of colour 0. Then for any $e:=\{a_{n_1},a_{n_2},\dots,a_{n_\ell}\}\in[M]^\ell$ where $n_1<n_2<\dots<n_\ell$, we have that $\{a_{n_2},\dots,a_{n_\ell}\}\in[A_{n_1+1}]^{\ell-1}$ hence $e\in[a_{n_1},A_{n_1+1}]$ and thus $c(e)=0$.

Hence M is c-monochromatic. □



Now we derive a finitary version from this using compactness:

Finite Ramsey Thm. For any $\ell,k\in\mathbb{N}^+$ and any $m\in\mathbb{N}^+$, there is $N\in\mathbb{N}$ such that for any colouring $c:[\bar N]^\ell\to\bar k$, where $\bar n:=\{0,1,\dots,n-1\}$ for $n\in\mathbb{N}$, there is a c-monochromatic subset $M\subseteq\bar N$ of m elements.

Proof. For notational convenience, we prove for $\ell:=2$ and $k:=2$. Suppose towards a contradiction that there is $m\in\mathbb{N}^+$ such that no matter how large $N\in\mathbb{N}$ we take, there is a "bad" colouring $c:[\bar N]^2\to\{0,1\}$ with no monochromatic subset of size m. Let $\sigma := (R_0, R_1)$, where R_0, R_1 are $\overset{?}{\ell}$-ary relations and let the σ-theory T be the conjunction of the following sentences:

(i) For $i = 0, 1$, $\varphi_i := \forall x \forall y \; R_i(x,y) \to [(x \neq y) \wedge \neg R_{1-i}(x,y)]$.

(ii) $\forall x \forall y \; [x \neq y \to (R_0(x,y) \vee R_1(x,y))]$.

(iii) $\forall x_1 \forall x_2 \ldots \forall x_m \left(\bigwedge_{1 \le i < j \le m} x_i \neq x_j \to \bigvee_{1 \le i_1 < j_1 \le m} \; \bigvee_{1 \le i_2 < j_2 \le m} [R_0(x_{i_1}, x_{j_1}) \wedge R_1(x_{i_2}, x_{j_2})] \right)$.

Then this T has arbitrarily large finite models by our hypothesis, so it has an infinite model (by compactness). But this infinite model will have an infinite monochromatic subset by Infinite Ramsey theorem, contradicting axiom (iii) which says that there is no monochromatic subset of size m. $\square$

Remark. The compactness-and-contradiction arguments are as follows: suppose there are arbitrarily large counter-examples to the desired statement, then there is an infinite counter-example (by compactness), which we have proven doesn't exist via other infinitary tools.

Completeness of theories and categoricity.

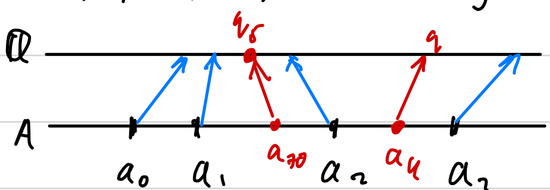
The following notion is critical in model theory and provides a tool for proving completeness (as well as quantifier elimination and other things).

Def. Let κ be a cardinal. Call a σ-theory T *κ-categorical* if any two models $\underline{A}, \underline{B} \models T$ of cardinality κ are isomorphic. We say that T is *categorical* if it is categorical in some cardinal κ.

Examples. (a) For a finite signature σ and any finite σ-structure $\underline{A}$, $\mathrm{Th}(\underline{A})$ is $|A|$-categorical. (This was a homework exercise)

(b) DLO is $\aleph_0$-categorical. Indeed, all ctbl dense linear orders w/o endpoints are isomorphic to $\underline{Q} := (\mathbb{Q}, <)$.

Proof. Let $\underline{A} \models DLO$ be ctbl model and we build an isomorphism $h: \underline{A} \xrightarrow{\sim} \underline{Q}$ by a back-and-forth argument. Enumerate $A = (a_n)_{n \in \mathbb{N}}$, $\mathbb{Q} = (q_n)_{n \in \mathbb{N}}$. We inductively build an increasing sequence (h_n) of partial order-isomorphisms $A \to \mathbb{Q}$ with finite domains s.t. $a_k \in dom(h_{2k})$ and $q_k \in im(h_{2k+1})$. Let $h_{-1} := \emptyset$, so $dom(h_{-1}) = \emptyset$, $im(h_{-1}) = \emptyset$. Suppose h_{2k-1} is defined. Define $h_{2k}: dom(h_{2k-1}) \cup \{a_k\} \to \mathbb{Q}$ by taking $h_{2k}|_{dom(h_{2k-1})} := h_{2k-1}$ and if $a_k \notin dom(h_{2k-1})$, then define $h_{2k}(a_k)$ so that h_{2k} is an order-isomorphism, which is possible because $\underline{Q}$ is dense linear order without endpoints (see the picture).



Similarly, we define h_{2k+1} extending h_{2k} and still an order-isomorphism but also $q_k \in im(h_{2k+1})$, see the picture. Treating the h_n as sets of pairs, we define $h := \bigcup_{n \in \mathbb{N}} h_n$, so h is an order-isomorphism with domain $= A$ and image $= \mathbb{Q}$, and thus an isomorphism from $\underline{A}$ to $\underline{Q}$. $\square$